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Exponential almost Sure Stabilization of Nonlinear Delay Differential Systems under Stochastic Optimal Control Driven by Ito Brownian Noise

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

This study investigates the role of Brownian white noise in stabilizing nonlinear optimal control delay differential equations $(OCDDE_s)$ that are typically unstable in their deterministic form. The technique applied involves the use of Lyapunov sample exponent and a specialized partial differential equation suggested by Mao, (1997). It is demonstrated that if the noise scaling parameters of the stochastically perturbed equation is finite, then the new stochastic optimal control delay differential equation $(SOCDDE_s)$ is self - stabilized in an almost sure exponential sense. This phenomenon does not occur in deterministic optimal control delay differential equations where noise is absent.

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1 Introduction

Delay differential equations are prevalent in various real - world scenarios. For examples: Delay differential equations are used to model the dynamics of control systems, the stability of mechanical structures, and the behavior of materials. In biological systems, they employed to model populations and the spread of diseases. A delay differential equation is a differential equation with deviating argument, this means, it is an equation that contains some functions and some of its derivatives to deviating argument values (Kolmanovskii & Myshkis, 1992).

When dynamical systems are subjected to performance criteria, aimed at achieving certain targets, the stability of such systems can form an area of interest to research on (Oguztoreli, 1979). Since (1892) when Lyapunov introduced the concept of stability into the study of dynamic system. Stability means insensitive of the system to little changes in its initial state or parameters of the system.

To understand stability, it is worthy to note that stability is of different types viz: mean square stability, stochastic stability, moment stability, almost sure exponential stability, stability in probability, asymptotic stability etc. Stamova & Stamov, (2013) obtained the stability of the zero solution of differential equations with maximum, by using Lyapunov functions and Razumikhin techniques. Xiao & Zhu, (2021) studied the stability of switched stochastic delay differential systems with unstable subsystems. Li et al., (2021) considered the stabilization of multi - weights stochastic complex networks with time delay driven by general Brownian motion. Zhu & Huang, (2021) established the stability for the class of stochastic delay nonlinear systems driven by general Brownian motion Ngoc, (2021) established the new criteria for the mean square exponential stability for the neutral stochastic functional differential equations. For a stable system, the trajectories which were close to each other at a specific instant should therefore remain close to each other at all subsequent instances (Mao, 1997). Shen et al., (2021) established the sufficient condition for the mean square exponential instability of stochastic differential equations by general Levy process with non lipschitz coefficients, their designed a discrete time feedback control in the drift part and obtained the mean square exponential stability and quasi - sure exponential stability for the controlled systems. Mao & Mao, (2017) studied the existence and uniqueness of solutions to neutral stochastic functional differential equations with Levy noise or Jumps. Chen et al., (2016) discussed the exponential stability for the neutral stochastic delay differential equations with time -varying delay. Liu, (2017) established a theory about the property of almost sure path-wise exponential stability for a class of stochastic neutral functional differential equations by developing a semi group scheme for the drift part of the systems under consideration and with path-wise stability through a perturbation approach rather than moment stability. Hasminskii, (1967) established the almost sure exponential stability of linear stochastic differential equations. It was established that, by the uses of Lyapunov function and the ideas of generalized moment inequalities as well as borel - Cantelli lemma under certain conditions on the drift and diffusion coefficients, p^{th} - moment, exponential stability implies the almost sure exponential stability Atonuje, (2015). Wei, (2019) studied the almost sure exponential stabilization of linear and nonlinear stochastic systems by stochastic feedback control with Levy noise from discrete time systems , using the techniques to generalized Ito formula for Levy stochastic integral , Borel - cantelli lemma , Burkholder - Davis - Gundy inequality , Holder inequality , and Gronwall inequality , the almost sure exponential stabilization of the linear and nonlinear stochastic systems was investigated and sufficient conditions were also provided. We shall consider a nonlinear n - dimensional ordinary delay differential equation of the form,

$$\dot{x}(t) = f(x(t), x(t-\tau), t);\ on\ t \geq t_0 \tag{1.1}$$

Where $f(.)\ is\ a$ volterra functional and τ is a constant time lag , for every initial value $x(t_0) = x_0 \in R^n)$ there exists a unique global solution which is denoted by $x(t, t_0, x_0)$. Suppose again that $f(0, 0, t) = 0$, for all $t \geq t_0$, equation (1.1) has the solution $x(t_0) = 0.$ This solution is called the trivial solution or equilibrium point . If equation (1.1) can be solved explicitly, it would be easier to determine whether the solution is stable. However, equation (1.1) can only be solved implicitly.

The inclusion of a stochastic ordinary delay differential equations. A stochastic delay differential equations have wide varieties of applications in many fields such as science, technology, physics, chemistry, structural (mechanics and optical bi- stability and fatigue cracking) financial mathematics, mathematical biology, radio astronomy and turbulent diffusion etc (Zhu et al., 2017).

A stochastic differential equation (SDEs) is an equation in which one or more of the terms is a stochastic process, resulting in a solution which is also a stochastic process. That is SDEs contain a variable which represents random white noise calculated as the derivative of Brownian motion. Different types of stochastic differential equations have been used to model different phenomena in various fields, such as unstable stock prices in finance (Merton, 1976). Atonuje et al., (2024) investigated the ability of a multiplicative Ito - type Brownian noise to stochastically stabilized the evolution of optimal control dynamical system with a volterra functional, described by an unstable nonlinear classical delay differential equation. The authors perturbed the equation by a multiplicative Brownian noise to form a stochastic optimal control system. By replacing the noise scaling parameter in the stochastic optimal control delay differential equation with a finite integral expression, the system become stochastically self stabilized in an almost sure exponential sense, under certain conditions and sufficiently small time delay. Kang & Jeong, (2021) studied the time optimal control to a target set for the semi- linear stochastic functional differential equations involving time delays under certain conditions on a target set and nonlinear terms, even when the equation contained unbounded principle operators. The approach was to construct a fundamental solution for corresponding linear system and established variation of constant formula for the given stochastic equation. Lanchares & Haddad, (2024) presented the framework addressed optimal nonlinear analysis and feedback control synthesis for nonlinear stochastic dynamical systems. They investigated the connections between stochastic Lyapunov theory and stochastic Hamilton - Jacobi - Bellman theory within a unified perspective. They also demonstrated that the closed - loop nonlinear systems asymptotically stable in probability was obtained through a Lyapunov function, which identified as the solution to the steady state to form the stochastic Hamilton Jacobi Bellman equation. This gurantees stochastic stability and optimal control systems. Wu et al., (2010) examined conditions under which the numerical solutions of stochastic delay differential equations may share the almost sure stability of exact solutions for non-linear stochastic delay differentia.

However, there are other types of random behavior that are possible, such as jump processes. Early work on (SDEs) was done to describe Brownian motion in Einsteins famous paper and at the same time by Smoluchowski. However, one of the earlier works related to Brownian motion is credited to Bachelier, (1901) in his thesis theory of speculation. This work was followed up on by Langevin. Later, Ito and Stratonovich put SDEs on more solid mathematical footing. Ito in 1994 laid the foundation of a stochastic calculus known today as the Ito calculus. This represents the stochastic generalization of the classical differential calculus, which models various phenomena in continuous time such as the dynamics of stock prices, physical systems or motion of a microscopic particle subjected to random fluctuations. The corresponding stochastic differential equations (SDEs) generalize the ordinary deterministic differential delay equation (ODDEs). Stochastic differential equations (SDE_S) are ordinary differential equations perturbed by Brownian noise.

This study have an important area of research, the stabilization of deterministic nonlinear optimal control delay differential systems under stochastic influence. Stability is of paramount importance to production managers, neural network experts, medical personnel, and social scientists. It is relevant in both theoretical and applied mathematics. This is so because an unstable system leads to poor production output. our improved knowledge of stabilization and instability can added to the progress of national economy.

In general, the first order stochastic nonlinear ordinary delay differential equation has the form;

$$\left.\begin{aligned} dX(t) = \ f(X(t), X(t-\sigma), t)dt + \mu g\big(X(t)\big)dB(t), t \geq 0 \\ x(t) = x(t_0) \end{aligned}\right\} \tag{1.2}$$

where $f(X(t)$ is called the drift function , $\mu g(X(t))$ is called the diffusion function and $B(t)$ is a Brownian noise, $B = \{B_t, t \geq 0\}$ defines the randomness of the physical systems and it is often called the white noise. The wiener process is the simplest intrinsic noise term that adequately model Brownian motion. The integral form of (1.2) is;

$$x(t) = x(t_0) + \int_0^t f(x(s), x(s-\sigma), s)ds + \int_0^t g\big(x(s)\big)dB(s), t \geq 0 \tag{1.3}$$

The first integral in (1.3) is a Volterra integral term and the second integral is an Ito stochastic integral with respect to the Brownian noise $B = \{B(t), t \geq 0\}$.

Let (Γ, ξ, p) be a complete probability space with filtration $\{\xi_t\}_{t_{\geq 0}}$. A standard one dimensional Brownian motion is a real valued continuous $\{\xi\}$ adapted process $\{B(t)\}_{t_{\geq 0}}$ which satisfies the following properties;

(i) $B_0 = 0$
(ii) The function $t \to B(t)$, is continuous
(iii) For $0 \leq s < t < \infty$, the increment $B_t - B_s$ is normally distributed with mean zero and variance $t - s$

2 Formulation of Optimal Control Delay Equations

Let (X) be a dynamic system whose state at time t is described by an $m-$dimensional vector $x(t) = (x_1(t), \dots, x_m(t)$. Assume that the system (X) is controlled by certain controllers, if these controllers are characterized at time t by an $n - dimensional$ vector $v(t) = (v_1(t), v_2(t) , \dots, v_n(t)$. Suppose Q is a compact and convex set in the $n - dimensional$ space A^n with points $v = (v_1, \dots, v_n)$,where $v(t)$ is called the control vector or the control function, Q is called the control region.

2.1 Admissible function

A measurable function $v(t)$ defined for $t \in [t_0, t]$, where $\dot{t} \in [t_1, t_2]$ is called admissible, if their range is in Q. Let E be the set of all admissible control. Suppose that (X) is the states corresponding to the time interval $I = [\propto, t_0]$,where $|\propto|$ is sufficiently large, is described by an $n - dimensional\ vector$;

$$Q(t) \in c([\propto, t_0], G) \tag{2.1}$$

where G is a compact region in the set A^n, containing the origin as an interior point. Given an $r - dimensional$ continuous real - valued function, the initial function ϕ , on the interval $\propto \leq t \leq t_0$ and $\phi(t_0)$, find a function $x(t)$, continuous for $t \geq \propto$ such that;

$$x(t) = \phi(t)\,, for\ \ \propto \leq t \leq t_0 \tag{2.2}$$

Let ϕ be the set of all allowable initial function $\phi(t) and\ E\ be\ the$ set of all allowable control vector $v(t)$,if $\dot{E}$ is a subset of the $n - dimensional$ space of points with co-ordinates $(v_1, v_{2\,,\dots,V_n)}$, suppose that the vector $v(t) \in \dot{E}$. Then the set $\dot{E}$ is called the control region where $\dot{E}$ is the range of all $v(t) \in E.$ Any element $\phi(t) \in \phi$ is said to be admissible initial function and any $v(t) \in E$ is said to be an admissible control. A pair $\{\phi, u\}$ with $\phi(t_0)\ \in \phi\ and\ v \in E$ is called admissible pair or admissible policy.

Let $r(t) = r(t, t_0, \phi, v)$ be a trajectory corresponding to an admissible pair $\{\phi(t_0), u(t)\} \in p$. suppose that;

a. The trajectories $r(t)$ which corresponding to admissible pairs $\{\phi(t_0), v(t)\}$ remain in a given compact region

   $G \in A^n\ for\ r(t) = \phi(t), t \in [\propto, t_0]$

b. The control region $\dot{E}$ is compact, convex and contains the origin of A^n as an interior point and the members of the set E of all admissible controls consist of all measurable functions defined for $t \in [t_0, t]$, [$\dot{t} \in t_1, t_2]$, whose range is contained in $\dot{E}$.

c. The region G is compact, convex and contains the origin of A^n as an interior point and the set ϕ of all admissible initial function which is defined for $\phi = \{\phi(t) \in c([\propto, t_0], G\}$, $\phi(t)$ is admissible, compact, convex and contains the origin as an interior point.

d. During the evolution of the process, the integral constraints
$\int_{t_0}^{t} r_i(r, v, t)dt \subseteq W_i \, , i = 1,2 \ldots n$

where w_i *is a closed subsets of* A^n

We shall consider the optimal controlled system (X) which is described by the nonlinear deterministic ordinary delay differential equation of the form

$$\left.\begin{aligned} x'(t) = f(x(t), v(t), x(t-r), t) \; t > 0) \\ x(t) = \phi(t), t \in [-\Gamma, 0] \end{aligned}\right\} \quad (2.3)$$

Where $F(.)$ is a volterra functional defined and bounded for $t \in [-\Gamma, 0]$ satisfying the following conditions (a)-(d) for all $v(t)\epsilon\phi$ *and* $\forall$ $x(t)\epsilon c(I[t_0, T], G)$, where G is a compact subset of an n-dimensional space $\dot{E}$, F is integrable and continuous for x *and* v , $r\epsilon(0,1)$ is a constant time lag or delay, $v(t)$ is the control vector, defined for $t \in [-\Gamma, 0]$ and $x(t) = \phi(t), t \in [-\Gamma, 0]$ with $\phi(t_0) \in \phi$ is the initial datum. By the solution of (2.3), we mean a continuous vector function $x(t) = \phi(t)$ such that $x(t)$ satisfies $Eq.$ (2.3) as well as the initial condition $x(t) = \phi(t), t \in [-\Gamma, 0], \phi(t_0)$. The solution of equation (2.3) is said to be unstable on [-Γ,0], if for every $\varepsilon > 0$ and for any t ≥ 0. Let $x(t\ t_0, x_0)$ be the solution of (2.3) satisfying $x(t) = \phi(t)$, where $t \geq t_0$ and $x_0 \in \mathbb{R}$ .Then $x(t) = \phi(t)$ is said to be unstable if for any ε > 0 and any $t \geq t_0$, there exists a $\delta = \delta(\varepsilon, t_0) < 0$ such that $\left| x(t_0) - \phi(t) \right| > 0 \Rightarrow \left| x(t, t_0, x_0) - \phi(t) \right| > \varepsilon \; \forall \; t \geq t_0$.

We perturbed Eq. (2.3) by an Ito-type multiplicative Brownian noise which resulted stochastic optimal control delay differential equation of the form;

$$\left.\begin{aligned} dX(t) = f(X(t), v(t), X(t-r), t)dt + \mu g(X(t), t)dB(t), t > 0 \\ x(t) = \phi(t) \, , t\epsilon[-\Gamma, 0] \end{aligned}\right\} \quad (2.4)$$

Where $f\epsilon L^1([t_0, T] \, , R^d)$, $g\epsilon L^2([t_0, T], R^{dn})$, $v(t)$ is the control vector , μ is the noise scaling parameter which determine the strength of fluctuation of the system , $f(.)$ is called the drift function ,$g(X(t)$ is called diffusion function , $r \in (0,1)$ *is called the constant time lag* and $B(t)$ is a Brownian motion , $B = \{B(t), t \geq 0\}$, defined on the probability triple $(\phi, \mathcal{F}, P)$ with filtration $\{X(t)\}_{t_{\geq 0}}$, defined on $\{\phi, \mathcal{F}, P\}$ satisfying (2.2) together with the initial function which is the same with that of the $OCDDE_S$ (2.3).

Definition 1 (The trivial solution of SOCDDE)

Assume that $\{X(t)\}_{t_{\geq 0}}$ is the solution of (2.2). Suppose that $f(0,0,0, t) \equiv 0, g(0,0,0) \equiv 0 \; \forall \; t > 0$. It follows that (2.2) has the solution $X(t) \equiv 0$ corresponding to the initial datum $X(t_0) \equiv 0$ is called the trivial or zero solution of (2.2).

Definition (2)

The zero solution $X(t, 0,0,0)$ of the SOCDDE (2.2) is said to be almost surely exponentially stable if $\lim_{t \to \infty} sup \frac{1}{t} log|X(t, t_0, x_0| < 0 \; \forall \; X_0 \in R^d$.

The following Assumptions, Lemmas and Theorems are based on the Ito formula is called exponential martingale inequality. it is useful to the proof of the main result.

Assumption (2.1)

H1: we assume that the following hypothesis hold

(i) $|X^T \Delta f(X, t) + X^T \Delta g(X, t) \leq H|X^T \Delta X|^2|$
(ii) Trace $(h^T(X, t)\Delta h(x, t) \leq \pi X^T \Delta X)$

(iii) $|h^T(X,t)\Delta h(x,t)|^2 \geq \gamma|X^T\Delta X|^2\ \forall\ t > 0\ , and\ x \in R^d$

3 Main Results

Lemma (3.1)

Suppose that assumption (H1) hold. Then the solution of equation (2.2) satisfies the property that $p\{x(t,x_0) \neq 0\ \forall\ t \geq 0\} = 1$ provided that $x_0 \neq 0$.

Proof

If lemma (3.1) is false , there exists some $x_0 \neq 0$ such that $p(\gamma < \infty) > 0$, where γ is the time of first reaching state zero that is $\gamma = \inf\{t \geq 0: x(t) = 0\}$ Let $x(t) = x(t,x_0)$. We can find $t^* > 0\ and\ \emptyset > 0$, large enough to ensure that $P(B) > 0$, where $B = \{\omega: \gamma \leq t^*\ and\ |x(t)| \leq \emptyset - 1\ \forall\ 0 \leq t \leq \gamma\}$ for every $0 < \varepsilon < |x_0|$. We have $\gamma_\varepsilon = \inf\{t \geq 0: |x(t)| \leq \varepsilon \mapsto |x(t)| \geq \emptyset\}$. By Ito formula we have, for every $0 \leq t \leq t^*$,

$$E[|x^T(t\wedge\gamma_\varepsilon)Hx(t\wedge\gamma_\varepsilon)|^{-1}] \leq x_0^T Hx_0 2E\int_0^{t\wedge\gamma_\varepsilon}|x^T(s)Hx(s)|^{-1}|x^T(s)Hf(x(s),s)|\mathrm{ds}$$

$$+4E\int_0^{t\wedge\gamma_\varepsilon}|x^T(s)Hx(s)|^{-3}x^T(s)Hg(x(s),s)\ |^2\left(\int_0^s|\beta(u)x(u)|^p du\right)^2 ds$$

from the (H1) above, we have

$$E[|x^T(t\wedge\gamma_\varepsilon)Hx(t\wedge\gamma_\varepsilon)|_1^{-1} \leq |x_0^T Hx_0|^{-1} + \delta E\int_0^t|x^T(s)Hx(s)|ds^{-1}$$

$$\leq|x_0^T Hx_0|^{-1} + \delta E\int_0^{t\wedge\gamma_\varepsilon}[|\ x^T(s\wedge\gamma_\varepsilon)Hx(s\wedge\gamma_\varepsilon)\ |]^{-1} ds$$

Where δ is a constant dependent but independent of ε . By the Gronwall inequality we have

$$E[|x^T(t^*\wedge\gamma_\varepsilon)Hx(t^*\wedge\gamma_\varepsilon)|^{-1}] \leq |x_0^T Hx_0|^{-1}e^{\gamma t^*} \tag{3.1.1}$$

Since $\omega\epsilon\beta\ , then\ \gamma_\varepsilon \leq t^*\ and\ |x(\gamma_\varepsilon)| = \varepsilon$. From the inequality (3.1.1) we have

$$(\varepsilon^2\|H\|)^{-1}p(\beta) \leq |x_0^T Hx_0|e^{\beta t^*} \tag{3.1.2}$$

as $\varepsilon \to 0, p(\beta) = 0$, hence contradicts the definition of β.

Theorem (3.2)

Consider the system

$$dX(t) = [f(X(t),V(t),X(t-\tau),t)]dt + \mu g(X(t),t)dB(t) \tag{3.2.1}$$

Suppose that there exists a function $\propto\in C^{2,1}(R^d \times [0,\infty),R)$, the family of all non negative function $\alpha(x,t)$ define on $R^d \times [0,\infty),R)$ such that they are continuously twice differentiable in $x\ and\ t\ and\ constants\ h > 0, k > 0, k_2 \in R, k_3 \geq 0$ such that $\forall\ x \neq 0\ , t \geq 0$,we have;

(a) $k_1|x|^h \leq \beta(x,t)$
(b) $L\beta(x,t) \leq k_2\beta(x,t)$
(c) $|\beta_x(x,t)g(x,t)|^2 \geq k_3\beta^2(x,t)$. Then

$$\lim_{t\to\infty} sup\frac{1}{t}log|x(t,t_0,x_0)| \leq -\frac{k_3-2k_2}{2h} \tag{3.2.2}$$

Almost surely $\forall\, x_0 \in R^d$. If in particular that $k_3 > 2k_2$, the solution of stochastic optimal control delay differential equation (3.2.1) is almost surely exponentially stable.

Proof,

If the sample Lyapunov exponents of the solution (3.2.2) holds for $x_0 = 0$, then $x(t, t_0, x_0) \equiv 0.$ We need to show (3.2.2) for $x_0 \neq 0$. suppose $x(t) = x(t, t_0, x_0)$ for any $x_0 \neq 0$. By Lemma (3.1) , $x(t) \neq 0 \,\forall\, t \geq 0$ almost surely . Apply the Ito formula and by condition (b) and (c) , we show that for $t \geq 0$,we have $\log f(x(t), t) \leq \log f(x_0, 0) + k_2(t - 0) + N(t) - \frac{1}{2}\int_0^t \left|\frac{\beta_x(x(s),s)g(x(s),s)}{\beta^2(x(s),s)}\right| ds^2$

Where $N(t) = \int_0^t \frac{\beta_x(x(s),s)g(x(s),s)}{\beta(x(s),s)} dB(s)$, is a continuous martingale with initial value $N(0) = 0$. If $\varepsilon \in (0,1)$ is arbitrary and Let $m = 1,2, \ldots$, By the exponential martingale inequality, we have;

$$p\left\{sup_{0\leq t\leq m}\left[N(t) - \frac{\varepsilon}{2}\int_0^t \left|\frac{\beta_x(x(s),s)g(x(s),s)}{\beta^2(x(s),s)}\right|^2 ds\right] > \frac{2}{\varepsilon} logm\right\}\frac{1}{m^2} \tag{3.2.3}$$

By the Borel-Cantelli Lemma, we observed that almost all $\rho\epsilon\varphi$,there is an integer $m_0 = m_0(\varphi)$ such that $m \geq m_0$

$$N(t) \leq \frac{2}{\varepsilon} logm + \frac{\varepsilon}{2}\int_0^t \left|\frac{\beta_x(x(s),s)g(x(s),s)}{\beta^2(x(s),s)}\right|^2 ds \tag{3.2.4}$$

Eq. (4.16) hold $\forall\, 0 \leq t \leq m$

$$\log f(x(t), t) \leq \log f(x_0, t_0) - \frac{1}{2}[(1 - \varepsilon)k_3 - 2k_2](t - 0) + \frac{2}{\varepsilon}\log m \tag{3.2.5}$$

$\forall\, 0 \leq t \leq m$, $m \geq 0$ almost surely. For almost all $\rho\epsilon\varphi$,let $m - 1 \leq t \leq m$ and $m \geq 0$, we have $\frac{1}{t}\log\beta(x(t), t) \leq -\frac{t-0}{2t}[(1 - \varepsilon)k_3 - 2k_2] + \frac{\log\beta(x_0,0)+\frac{2}{\varepsilon}\log m}{m-1} \mapsto$

$\lim_{t\to\infty} sup\frac{1}{t}\log\beta(x(t), t) \leq -\frac{1}{2}[(1 - \varepsilon)k_3 - 2k_2]$, By condition (a) we have $\lim_{t\to\infty} sup\frac{1}{t}\log|x(t)| \leq -\frac{(1-\varepsilon)k_3-2k_2}{2h}$, since $\varepsilon > 0$ is arbitrary. Hence the stochastic optimal control delay differential system is almost surely exponentially stable.

In the present research, we established the stochastic self - stabilization of a nonlinear deterministic optimal control system with time delay. The system of ordinary non-linear deterministic optimal control delay differential equation were perturbed by the addition of a multiplicative white noise of an Ito-type represented by the Brownian motion , which resulted in the stochastic optimal control delay differential equation ($SOCDDE$) that is (2.4) , if the average impact of the noise scaling parameter μ is kept finite by making it as large as possible , the $SOCDDE$ is stable in an almost sure exponential sense . Our techniques involves the use of Lyapunov sample exponent and stochastic perturbation.

4 Conclusion

In this study, we established the almost sure exponential stability of the non-linear stochastic optimal control differential equations ($SOCDDE_S$) with a constant delay or time lag. Our findings reveal that, it is possible to stabilize an unstable dynamical systems with Ito - type Brownian noise. The sampled Lyapunov exponent must always be finite for the resulting stochastic system to be stabilized by Brownian noise. If the noise term is sufficiently large enough, the diffusion function is absent, the deterministic system will continue to be unstable. In particular, the new theorem enable us to stabilize unstable deterministic system in an almost sure exponential sense. The method of Lyapunov sample exponents together with stochastic perturbation were used to achieved stability result.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

Competing Interests

Authors have declared that no competing interests exist.

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